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## A Postscript on Steadfast Rings and $R$ -Algebras in One Variable

ELOISE HAMANN

*Elmhurst College, Elmhurst, Illinois 60126**Communicated by Richard G. Swan*

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Swan's paper, "On Seminormality" neatly characterizes commutative reduced steadfast rings as the reduced rings which are  $p$ -seminormal for all  $p$  [10]. Swan conjectured that  $R$ , a commutative ring, is steadfast iff  $R_{\text{red}}$  is steadfast.  $R_{\text{red}} = R/N$ , where  $N$  is the ideal of nilpotents of  $R$ . It is the purpose of this note to prove this result and to similarly extend some results of Greither on  $R$ -algebras in one variable.<sup>1</sup> A general lifting theorem is employed which may be of independent interest. All rings are commutative with 1.

### 1. BACKGROUND AND DEFINITIONS

If  $R$  and  $S$  are isomorphic rings, then  $R[X]$  is clearly isomorphic to  $S[Y]$ , where  $X$  and  $Y$  are indeterminates. If  $R[X_n]$  is isomorphic to  $S[Y_n]$  where  $X_n$  and  $Y_n$  are collections of  $n$  independent indeterminates, then  $R$  and  $S$  are called stably isomorphic. The question of whether  $R$  and  $S$  stably isomorphic implies  $R$  and  $S$  isomorphic was settled by Hochster in the negative as recently as 1972 [6]. Subsequently, a number of people sought conditions on  $R$  so that  $S$  stably isomorphic to  $R$  implies  $S$  isomorphic to  $R$ . When the latter condition holds,  $R$  is called invariant. Invariant rings include fields  $K$ ,  $K[X]$ ,  $K[X, Y]$ , and certain one dimensional domains. [1–3, 7]. (The problem under discussion here should not be confused with the related Zariski problem of whether  $K(X)$  isomorphic to  $L(Y)$  implies  $K$  and  $L$  isomorphic where  $K, L$  are fields and  $X, Y$  are field indeterminates.)

Related to the notion of invariance is the idea of  $R$ -invariance. An  $R$ -algebra  $A$  is  $R$ -invariant if  $A[X_n] \cong_R B[Y_n]$  implies that  $B$  is  $R$ -isomorphic to  $A$ . Since  $R$  itself is easily seen to be  $R$ -invariant, the simplest ring one might

<sup>1</sup> Since writing the paper it has come to my attention that Asanuma has proved the mentioned result in his paper "On Stably Polynomial Algebras."

investigate for  $R$ -invariance is  $R[X_1]$ .  $R$  is called steadfast if  $R[X_1]$  is  $R$ -invariant. Since it is natural to identify isomorphic rings,  $R$  is steadfast if  $R[X_{n+1}] = S[Y_n]$  with  $R \subseteq S$  implies that  $S \cong R[X_1]$ . The following unexpected example of a nonsteadfast ring depends heavily on the failure of  $R$  to be seminormal.

Let  $R = \mathbb{Z}/p\mathbb{Z}[a^2, a^3] \subseteq \mathbb{Z}/p\mathbb{Z}[a]$ , or let  $R = \mathbb{Z}[pa, a^2, a^3] \subseteq \mathbb{Z}[a]$ , where  $a$  is an indeterminate. Let  $B = R[Z - aZ^p, Z^p]$ , where  $Z$  is an indeterminate independent of  $a$ . Then  $R[X_1, X_2] = B[Y_1]$  if  $X_1 = Z - a(Z + aY_1)^p$  and  $X_2 = Y_1 + (Z + aY_1)^p$ . The relation  $X_1 + aX_2 = Z + aY_1$  allows one to solve for  $Y_1, Z^p$ , and  $Z - aZ^p$  in terms of  $X_0$  and  $X_1$  [5, p. 14].

$R$  is defective in the sense that there is a "missing element." There are a number of characterizations of the seminormal property, one of which is that there are no "missing elements." More precisely, Traverso showed the following are equivalent under the assumption that the integral closure of  $R$  in its total quotient ring  $Q(R)$  is finite over  $R$ .

(1)  $R$  is seminormal, i.e.,  $R = +R = \{X \in Q(R) \mid X/1 \in R_p + J(R_p Q(R)) \text{ for all primes } P \text{ of } R\}$   $J(\ ) = \text{Jacobson radical}$ .

(2)  $\text{Pic } R = \text{Pic } R[X]$ , where  $\text{Pic}$  stands for the Picard group of isomorphism classes of finitely generated projective rank one modules under  $\otimes_R$ .

The author, however, prefers to think of  $R$  as seminormal if:

(3)  $a^2, a^3 \in R$  implies  $a \in R$ , where  $a \in Q(R)$ , and in fact this condition is equivalent to (1) and (2) [5, p. 14, 15].

The failure of (3) figures prominently in the failure of  $R$  above to be steadfast. However, the condition that  $\text{Pic } R \cong \text{Pic } R[X_1]$  is most useful in proving that a ring  $R$  is steadfast.

Swan generalized the notion of seminormality by removing the finiteness assumption and reformulating (3) as (3')  $b, c \in R$  with  $b^3 = c^2$  implies there exists  $a \in R$  with  $a^2 = b, a^3 = c$ . A ring  $R$  is called seminormal if  $R$  is reduced and (3') holds. Swan established the equivalence of (1) with this definition if  $R$  is reduced. Swan also defined  $p$ -seminormality as follows.  $R$  is  $p$ -seminormal if  $R$  is reduced and  $b, c, d \in R$  with  $b^3 = c^2, d^2 = p^2b, d^3 = p^3c$ , implies there exists an  $a \in R$  with  $a^2 = b, a^3 = c$ , and  $pa = d$ . He obtains results relating the  $p$ -seminormality of  $R_{\text{red}}$  and the Picard groups of  $R$  and  $R[X_n]$ . These results were used to tie up the loose ends of [5] in the result mentioned in the opening paragraph. Thus, the settling of the conjecture that  $R$  is steadfast iff  $R_{\text{red}}$  is steadfast means that steadfast rings will be well characterized.

## 2. A LIFTING THEOREM AND THE RESULT

LEMMA 2.1. *Let  $R$  be any ring,  $\bar{R} = R/N$ , where  $N$  is the ideal of nilpotents. Let  $\bar{S}$  be a finite set of generators of  $\bar{R}[X_n]$  over  $\bar{R}$ . Then if  $S$  is any set of lifts of  $\bar{S}$  to  $R[X_n]$ ,  $R[S] = R[X_n]$ . In particular, if  $\bar{B}[\bar{Y}_m] = \bar{R}[X_n]$  with  $B \supseteq \bar{R}$  and finitely generated over  $\bar{R}$ , then  $B[Y_m] = R[X_n]$ , where  $B$  is any lift of  $\bar{B}$  and  $Y_i$  lifts  $\bar{Y}_i$ .*

*Proof.* Since  $\bar{R}[\bar{S}] = \bar{R}[X_n]$  and  $\bar{S}$  is finite, there is a finite set of coefficients from  $\bar{R}$  which relate  $X_n$  and  $\bar{S}$ . Let  $\bar{R}_0$  = the subring of  $\bar{R}$  generated by 1 and these coefficients. Let  $R_0$  be a lift of  $\bar{R}_0$  which contains the coefficients required to obtain the elements of  $S$ .  $R_0$  can also be chosen as a finitely generated ring and is hence Noetherian. We have  $R_0[X_n] = R_0[S] + N_0 R_0[X_n]$  where  $N_0$  is the ideal of nilpotents of  $R_0$ . Since  $N_0^k = 0$  for some  $k$ ,  $R_0[X_n] = R_0[S] + N_0^2 R_0[X_n] = \dots = R_0[S]$ . Tensor by  $R$  over  $R_0$  to obtain  $R[X_n] = R[S]$ . The rest is clear.

Arguments similar to the one given above will be given at several points in the paper, and will only be outlined.

The following theorem in its present form is due to Swan. It is a generalization of a result in an earlier version of this paper.

THEOREM 2.2. *Let  $R$  be a commutative ring,  $\bar{R} = R_{\text{red}} = R/N$  and let  $\bar{\phi}: \bar{R}[X_1, \dots, X_n] \rightarrow \bar{R}[X_1, \dots, X_n]$  be an  $\bar{R}$ -algebra homomorphism with  $\bar{\phi}^2 = \bar{\phi}$ . Then  $\phi$  lifts to an  $R$ -algebra homomorphism  $\phi: R[X_1, \dots, X_n] \rightarrow R[X_1, \dots, X_n]$  with  $\phi^2 = \phi$ .*

*Proof.* Since  $\bar{B} = \bar{\phi}(\bar{R}[X_1, \dots, X_n])$  is finitely generated over  $\bar{R}$ , we can reduce to the case where  $R$  is Noetherian. (Choose  $\bar{R}_0$  as the subring of  $\bar{R}$  generated by 1, and the finite set of coefficients involved in generating  $\bar{B}$  over  $\bar{R}$ . Choose  $R_0$  as a Noetherian lift of  $\bar{R}_0$ . If  $\phi_0: R_0[X_1, \dots, X_n] \rightarrow R_0[X_1, \dots, X_n]$  is such that  $\phi_0^2 = \phi_0$ , then tensoring by  $R$  yields  $\phi: R[X_1, \dots, X_n] \rightarrow R[X_1, \dots, X_n]$  with  $\phi^2 = \phi$ .)

Now lift  $\bar{\phi}$  to an  $R$ -algebra homomorphism  $\phi: R[X_1, \dots, X_n]$ . Let  $B = \text{Im } \phi$ . It is clear that  $B$  maps onto  $\bar{A} = \text{Im } \bar{\phi}$  so we have

$$\begin{array}{ccccc} R[X_1, \dots, X_n] & \xrightarrow{p} & B & \xhookrightarrow{j} & R[X_1, \dots, X_n] \\ \downarrow & & \downarrow & & \downarrow \\ \bar{R}[X_1, \dots, X_n] & \xrightarrow{\bar{p}} & \bar{A} & \xhookrightarrow{j} & \bar{R}[X_1, \dots, X_n] \end{array}$$

Let  $\theta = pj: B \rightarrow B$ . Apply  $\bar{R} \otimes_R$  to the top row, obtaining

$$\bar{R}[X_1, \dots, X_n] \xrightarrow{\bar{\theta}} \bar{B} \xrightarrow{j} \bar{R}[X_1, \dots, X_n],$$

with  $\bar{B} = B/NB$  ( $\tilde{f}$  may not be injective). Since  $\tilde{f}\bar{p} = \tilde{f}\bar{p} = \bar{\phi}$ ,  $\tilde{f}$  factors as  $\bar{B} \xrightarrow{q} \bar{A} \hookrightarrow \bar{R}[X_1, \dots, X_n]$  and  $\bar{p} = q\tilde{p}$ .  $\theta$  induces  $\bar{\theta} = \tilde{p}\tilde{f}: \bar{B} \rightarrow \bar{B}$  and  $\bar{\theta}^2 = \bar{\theta}$ , since  $\bar{\theta}^2 = \tilde{p}\tilde{f}\tilde{p}\tilde{f} = \tilde{p}\tilde{f}q\tilde{p}\tilde{f}q = \tilde{p}\tilde{f}\tilde{p}\tilde{f}q = \tilde{p}\tilde{f}q = \tilde{p}\tilde{f} = \bar{\theta}$ , since  $\tilde{p}\tilde{f} = 1_{\bar{A}}$ . It follows that  $\theta^2 - \theta$  maps  $B$  into  $NB$ , and therefore maps  $N^i B$  into  $N^{i+1}B$ . If  $r$  is large,  $(\theta^2 - \theta)^r: B \rightarrow N^r B = 0$ , so  $(\theta^2 - \theta)^r = 0$ . Expansion shows that  $\theta^r$  is a linear combination of  $\theta^{r+1}, \theta^{r+2}, \dots, \theta^{2r}$ , so  $\theta^r B \subset \theta^{r+1}B$ , and therefore  $\theta^r B = \theta^{r+1}B$ . Since  $\theta: \theta^r B \rightarrow \theta^{r+1}B = \theta^r B$  is onto and  $\theta^r B$  is Noetherian,  $\theta$  restricted to  $\theta^r B$  is an automorphism. Let  $\alpha$  be the inverse of  $\theta^{r+1}$  restricted to  $\theta^r B$ , and define  $\phi$  to be the composition of the maps below.

$$R[X_1, \dots, X_n] \xrightarrow{p} B \xrightarrow{\theta^r} \theta^r B \xrightarrow{\alpha} \theta^r B \xhookrightarrow{j} R[X_1, \dots, X_n].$$

Then  $\phi^2 = j\alpha\theta^r p j\alpha\theta^r p = j\alpha\theta^{r+1}\alpha\theta^r p = j\alpha\theta^r p = \phi$ .

Since  $\theta$  induces  $1_{\bar{A}}$  on  $\bar{A} = B_{\text{red}}$ ,  $\bar{\theta}^r \bar{B} = \bar{A}$  and  $\bar{\alpha} = 1$ , so

$$\begin{array}{ccccccc} R[X_1, \dots, X_n] & \xrightarrow{p} & B & \xrightarrow{\theta^r} & \theta^r B & \xrightarrow{\alpha} & \theta^r B \xhookrightarrow{j} R[X_1, \dots, X_n] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{R}[X_1, \dots, X_n] & \xrightarrow{\bar{p}} & \bar{A} & = & \bar{A} & = & \bar{A} \xhookrightarrow{\bar{j}} \bar{R}[X_1, \dots, X_n] \end{array}$$

shows that  $\phi$  lifts  $\bar{\phi}$  (and also that  $A = \theta^r B$ ).

It is known that  $R/N$  steadfast implies  $R$  steadfast. The converse is one of the goals of this paper.

**PROPOSITION 2.3.** *If  $R_{\text{red}}$  is not steadfast, then  $R$  is not steadfast.*

*Proof.*  $R_{\text{red}}$  cannot be  $p$ -seminormal by Swan's result. Therefore, there exist  $\bar{b}, \bar{c}, \bar{d}$  in  $R_{\text{red}}$ , so  $\bar{b}^2 = \bar{c}^3$ ,  $p^3 \bar{b} = \bar{d}^3$ , and  $p^2 \bar{c} = \bar{d}^2$ , but no  $\bar{a} \in R_{\text{red}}$  such that  $\bar{a}^2 = \bar{c}$ ,  $\bar{a}^3 = \bar{b}$ ,  $p\bar{a} = \bar{d}$ .  $R_{\text{red}}[\bar{a}] = R_{\text{red}}[X]/(X^2 - \bar{c}, X^3 - \bar{b}, pX - \bar{d})$ , with  $\bar{a} = \bar{X}$  is such that  $R_{\text{red}}$  is injected into  $R_{\text{red}}[\bar{a}]$  [10, Lemma 4.3]. Therefore, we can construct the ring  $\bar{B}$  as in Section 1. Namely, let  $\bar{B} = R_{\text{red}}[Z - \bar{a}Z^p, Z^p] \subseteq R_{\text{red}}[\bar{a}][Z]$ .  $\bar{B}[y_1] = R_{\text{red}}[x_1, x_2]$ , where  $x_1 = Z - \bar{a}(Z + \bar{a}y_1)^p$  and  $x_2 = y_1 + (Z + \bar{a}y_1)^p$ . Note that  $x_1 + \bar{a}x_2 = Z + \bar{a}y_1$ ,  $y_1 = x_2 - (x_1 + \bar{a}x_2)^p$ ,  $Z^p = [x_1 + \bar{a}(x_1 + \bar{a}x_2)^p]^p$ , and  $Z - \bar{a}Z^p = x_1 + \bar{a}(x_1 + \bar{a}x_2)^p - \bar{a}[x_1 + \bar{a}(x_1 + \bar{a}x_2)^p]^p$ . While  $\bar{a} \notin R_{\text{red}}$ ,  $p\bar{a}, \bar{a}^i (i \geq 2)$  are in  $R_{\text{red}}$ , so all expressions have coefficients in  $R_{\text{red}}$  after high school algebra work. If  $\bar{B}$  were isomorphic to  $R_{\text{red}}[x_1]$  it could be argued that  $\bar{a} \in R_{\text{red}}$ .

Let  $\bar{\phi}: \bar{R}[X_1, X_2] \rightarrow \bar{R}[X_1, X_2]$  be defined as the identity on  $\bar{B}$ , and send  $y_1$  to 0. Then  $\bar{\phi}^2 = \bar{\phi}$ . Let  $\phi$  lift  $\bar{\phi}$  as in Theorem 2.2, so  $\phi: R[X_1, X_2] \rightarrow R[X_1, X_2]$ . Let  $B = \phi(R[X_1, X_2])$ . Let  $Y_1$  be a lift of  $y_1$ . By Lemma 2.1,  $B[Y_1] = R[X_1, X_2]$ . If  $\phi(Y_1) \neq 0$ , replace  $Y_1$  with  $Y_1 - \phi(Y_1)$ . By the idempotency of  $\phi$  we can assume  $\phi(Y_1) = 0$ .  $Y_1$  is not a zero divisor as it lifts  $y_1$ , which has invertible  $X_2$  coefficient. (Check the relations between  $X_1, X_2$  and

$Z - aZ^p, Z^p, y_1$ .) Thus, if  $Y_1$  is not an indeterminate over  $B$ , then  $b_0 + \cdots + b_n Y_1^n = 0$ , where  $b_0 \neq 0$ . However,  $\phi(b_0 + \cdots + b_n Y_1^n) = 0$  implies  $\phi(b_0) = b_0 = 0$ . This contradiction shows  $B$  is stably isomorphic to  $R[X]$ . If  $B$  were isomorphic to  $R[X]$ , then  $B_{\text{red}} = \bar{B}$  would be isomorphic to  $\bar{R}[X]$ . Thus,  $R$  is not steadfast.

We obtain the following theorem.

**THEOREM 2.4.**  *$R$  is steadfast iff  $R_{\text{red}}$  is steadfast iff  $R_{\text{red}}$  is  $p$ -seminormal for all  $p$ .*

### 3. A CONCRETE VIEW OF $B$

This section will only be of interest to those who are curious about the nature of  $B$  stably isomorphic to  $R[X_1]$ , but not isomorphic to  $R[X_1]$ . First suppose that  $R$  is reduced. If there is such a  $B$ , of course  $R$  is not steadfast and hence not  $p$ -seminormal for some  $p$ . We have the following proposition about  $B$ .

**PROPOSITION 3.1.** *If  $R$  is reduced and  $R[X_{n+1}] = B[Y_n]$  with  $B \not\cong R[X_1]$ , then there exist a finite number of elements  $a_1, \dots, a_k$  in the total quotient ring of  $R$  so that the following hold.*

- (1)  $B[a_1, \dots, a_k] \cong R[a_1, \dots, a_k][X_1]$ .
- (2) If  $B[a_1, \dots, a_k] = R[a_1, \dots, a_k][Z] = T$ , then there exists  $N$  so that  $f(Z)^N \in B$  and  $Nf(Z) \in B$  for all  $f(Z) \in T$ . In particular,  $Z^N \in B$  and  $NZ \in B$ .
- (3) There is a polynomial  $f(Z)$  in  $B$  of the form  $Z + r_2^* Z^2 + \cdots + r_j^* Z^j$  with  $r_i^* \in R[a_1, \dots, a_k]$ .

*Proof of (1).*  $B$  is finitely generated over  $R$  since  $B$  is isomorphic to  $R[X_{n+1}]/(Y_1, \dots, Y_n)$ . Thus, similar to the argument for 2.1, there exists  $R_0, B_0$  so  $R_0[X_{n+1}] = B_0[Y_n]$ , where  $R \cong R \otimes_{R_0} R_0$ , and  $B \cong R \otimes_{R_0} B_0$ . If  $B_0$  were  $\cong R_0[X]$ , we would obtain  $B \cong R[X]$ . Thus,  $R_0$  is not steadfast, and hence not  $p$ -seminormal for some  $p$ .  $R_0$  is reduced, Noetherian, and pseudogeometric. Therefore,  $R_0$  has finite normalization,  $\bar{R}_0$ .

Since  $R_0$  is not  $p$ -seminormal for all  $p$ , there exists  $a_1 \in \bar{R}_0 - R_0$  such that  $a_1^2, a_1^3, p_1 a_1 \in R_0$  for some  $p_1$  prime. If  $R_0[a_1]$  is  $p$ -seminormal for all  $p$ , we stop, as  $R_0[a_1]$  would be steadfast. Then  $B[a_1] \cong R[a_1][X_1]$ . If  $R_0[a_1]$  is not  $p$ -seminormal for all  $p$ , then there exists  $a_2 \in \bar{R}_0 - R_0[a_1]$  with  $p_2 a_2, a_2^2, a_2^3 \in R_0[a_1]$ . Since  $\bar{R}_0$  is a finite module over  $R_0$ , which is  $p$ -seminormal for all  $p$ , after a finite number of steps we must reach a ring  $R_0[a_1, \dots, a_k]$ , which is  $p$ -seminormal for all  $p$ . Further  $a_i^2, a_i^3, p_i a_i \in R_0[a_1, \dots, a_{i-1}]$ . Thus,  $B[a_1, \dots, a_k] \cong R[a_1, \dots, a_k][X_1]$ .

*Proof of (2).* Suppose  $B[a_1, \dots, a_k] = R[a_1, \dots, a_k, Z]$ , where  $\{a_1, \dots, a_k\}$  are chosen as in the proof of 1). Then  $Z \in R[a_1, \dots, a_k][X_{n+1}]$ . If  $S_i = R[a_1, \dots, a_i]$ , note that  $p_i S_i \subseteq S_{i-1}$ . Thus,  $p_1 p_2 \cdots p_k S_k \subseteq R$ . In addition,  $S_i^{p_i} = \{s^{p_i} | s \in S_i\} \subseteq S_{i-1}$ . Thus,  $S_k^N \subseteq R$ , and  $NS_k \subseteq R$  if  $N = p_1 p_2 \cdots p_k$ .

Similarly, if  $T_i = B[a_1, \dots, a_i]$ ,  $p_i T_i \subseteq T_{i-1}$  and  $T_i^{p_i} \subseteq T_{i-1}$ . Likewise  $T_k^N \subseteq B$  and  $NT_k \subseteq B$ . The result follows.

*Proof of (3).*  $B$  consists of polynomials  $p(Z)$  with coefficients in  $R[a_1, \dots, a_k]$  such that after substituting  $Z = Z(X_1, \dots, X_{n+1})$ , the resulting coefficients are in  $R$ . We assume  $\{Z, Y_n\}$  are without constants in  $R[a_1, \dots, a_k]$ .

*Step One.* If  $p(Z) = r_1^* Z + \text{higher terms}$  is in  $B$ ,  $r_1^* \in R$ . To see this, observe that  $\{Y_n, Z\}$  is a minimal generating set of  $R[a_1, \dots, a_k][X_{n+1}]$  over  $R[a_1, \dots, a_k]$ . Thus, the Jacobian matrix  $(\partial\{Y_n, Z\})/(\partial\{X_{n+1}\})$  is invertible. By multiplying  $Z$  by a suitable constant, we can assume the determinant is one. If  $Z = r_{1n+1}X_1 + \cdots + r_{n+1n+1}X_{n+1} + \text{higher terms}$ , we find by setting all  $X_i = 0$  in the Jacobian matrix that  $r_{1n+1}\hat{A}_{1n+1} + \cdots + r_{n+1n+1}\hat{A}_{n+1n+1} = 1$ , where  $\hat{A}_{ij}$  is the  $ij$  cofactor. Because  $\{Y_1, \dots, Y_n\} \subseteq R[X_n]$ ,  $\hat{A}_{ij} \in R$  if  $j = n+1$ . Now  $r_1^* Z + \cdots + r_m^* Z^m \in B$  implies  $r_1^* r_{in+1} \in R$  for all  $i$ , since it is ultimate coefficient of  $X_i$ . Thus,  $r_1^* = r_1^* r_{1n+1}\hat{A}_{1n+1} + \cdots + r_1^* r_{n+1n+1}\hat{A}_{n+1n+1} \in R$ .

*Step Two.* Define  $I$  to be set of elements of  $R[a_1, \dots, a_k]$ , such that  $i \in I$  if there is a polynomial  $p(Z) \in B$  with  $Z$  coefficient  $= i$ . By Step one and the preceding remark,  $I$  is an ideal of  $R$ . We show  $IR[a_1, \dots, a_k] = R[a_1, \dots, a_k]$  implies  $I = R$ . Therefore, suppose  $\sum_{j=1}^t i_j w_j = 1$ , where  $w_j \in R[a_1, \dots, a_k]$ , and  $i_j \in I$ . Let  $p_i$  and  $S_i$  be as in the Proof of 2. Then each term of  $(\sum i_j w_j)^{p_i} \in IR[a_1, \dots, a_{k-1}]$ , and by induction it follows that each term of  $(\sum i_j w_j)^N$  with  $N = p_1 p_2 \cdots p_k$  is in  $IR$ . Since  $1^N = 1$ ,  $I = R$ .

*Step Three.*  $I = R$ . If not, then  $IR[a_1, \dots, a_k] \neq R[a_1, \dots, a_k]$ , so  $IR[a_1, \dots, a_k] \cap R = K$  is a proper ideal of  $R \supseteq I$ . Let  $J =$  the ideal of  $U = R[X_{n+1}]$  generated by  $K$  and all  $X_i X_j$ .  $U/J = R/K + R/KX_1 + \cdots + R/KX_{n+1}$  as an  $R/K$  module. But  $U$  also  $= B[Y_n]$ , and each element of  $B$  is of the form  $r_0 + i_1 Z + r_2^* Z^2 + \cdots + r_n^* Z^n$ , with  $r_0 \in R$ ,  $i_1 \in I$ ,  $r_i^* \in R[a_1, \dots, a_k]$ . Since all coefficients end up in  $R$  after expressing the element in terms of the  $X_i$ , we get  $B \subseteq R + KX_1 + KX_2 + \cdots + KX_{n+1} + J \subseteq R + JU$ . In this case,  $U/J$  has only  $n+1$  generators. This contradiction implies  $I = R$ , and the result follows.

The example mentioned earlier in the paper is thus seen to be the simplest typical example. The following slightly more general example may be of some interest.

EXAMPLE. Let  $R$  be any ring  $\subseteq S$  with  $a \in S - R$ , but  $a^2, a^3, pa \in R$ . Let  $B = R[Z - af(Z^{p^n}), Z^{p^n}]$ , where  $f$  is a polynomial in  $R[X]$ .

$$B[Y_1] = R[X_1, X_2],$$

with

$$\begin{aligned} X_1 &= Z - af((Z + aY_1)^{p^n}), \\ X_2 &= Y_1 + f((Z + aY_1)^{p^n}), \\ X_1 + aX_2 &= Z + aY_1, \end{aligned}$$

so

$$\begin{aligned} Y_1 &= X_2 - f((X_1 + aX_2)^{p^n}), \\ Z^{p^n} &= [X_1 + af((X_1 + aX_2)^{p^n})]^{p^n}, \\ a^j Z^i &= a^j (X_1 + af((X_1 + aX_2)^{p^n}))^i, \quad j \geq 2, i \geq 1, \end{aligned}$$

and  $Z - af(Z^{p^n}) = X_1 - g(a^j Z^i, Y_i)$  for some  $g$ .

Swan communicated to me an explicit description of a  $B$  stably isomorphic to  $R[X]$  but not isomorphic to  $R[X]$  with  $R \subseteq B$ , in the case that  $R$  is not reduced.

PROPOSITION 3.2 (Swan). *If  $R$  is not steadfast and possibly not reduced, let  $b, c, d \in R$  with  $b^3 \equiv c^3$ ,  $p^2 b \equiv d^2$ ,  $p^3 c \equiv d^3 \pmod{\text{nil}(R)}$ , define  $a_{2n} = b^n$  for  $n \geq 1$ ,  $a_{2n+1} = b^n c$  for  $n \geq 0$ . Let  $F(T, U) = T^p + dT^{p-1}U + \binom{p}{2}a_2 U^2 T^{p-2} + \cdots + pa_{p-1} U^{p-1}T + a_p U^p - U \in R[T, U]$ , with  $T, U$  indeterminates. Note that if  $S = R_{\text{red}}[\bar{a}]$ , where  $\bar{a}^2 = \bar{b}$ ,  $\bar{a}^3 = \bar{c}$ ,  $p\bar{a} = \bar{d}$ ,  $\bar{a}_m = \bar{a}^m$ , then  $\bar{F}(T, U) = (T + \bar{a}U)^p - U$  in  $S[T, U]$ . Let  $B = R[T, U]/(F(T, U))$ . Then  $B$  is stably isomorphic not isomorphic to  $R[X]$ .*

*Proof.* Send  $B \rightarrow \bar{B} = \bar{R}[Z - \bar{a}Z^p, Z^p]$  by  $T \rightarrow \bar{t} = Z - \bar{a}Z^p$ ,  $U \rightarrow \bar{u} = Z^p$ .<sup>2</sup> Define  $\phi: R[X_1, X_2] \rightarrow B[Y_1]$  by  $X_1 \rightarrow$  lift of  $Z - a(Z + ay_1)^p$ ,  $X_2 \rightarrow$  lift of  $y_1 + (Z + ay_1)^p$ . Then

- (1)  $B_{\text{red}} \cong \bar{B} = \bar{R}[Z - \bar{a}Z^p, Z^p]$ .
- (2)  $\phi: R[X_1, X_2] \cong B[Y_1]$ .

<sup>2</sup> The reader may need to refer to the relations involved in  $\bar{B}[y_1] = \bar{R}[x_1, x_2]$  given in the proof of 2.3.

*Proof of (1).* Since  $\bar{F}(\bar{t}, \bar{u}) = 0$  and  $\bar{B}$  is reduced, we have

$$B \rightarrow B/(B \text{ nil } R) = \bar{R}[T, U]/(F(T, U)) \xrightarrow[\theta]{\begin{matrix} \nearrow B_{\text{red}} \\ \searrow \end{matrix}} \bar{B}.$$

Suppose  $G(T, U) \in \bar{R}[T, U]/(\bar{F}(T, U))$  goes to 0 in  $\bar{B}$ . Since  $\bar{F} = T^p + \dots$  is monic in  $T$ , we can change  $G \bmod \bar{F}$  to  $G = U^m \sum_{i=0}^{p-1} G_i(U) T^i$ , where  $U \nmid G_i(U)$  for some  $i$  (assuming  $G \neq 0 \bmod \bar{F}$ ). Since  $U \rightarrow Z^p$ , which is regular in  $\bar{B}$ ,  $\sum_{i=0}^{p-1} G_i(U) T^i \rightarrow 0 = \sum_{i=0}^{p-1} G_i(Z^p)(Z - \bar{a}Z^p)^i \in S[Z]$ . Let  $Z^p = 0$ . This yields  $\sum_{i=0}^{p-1} G_i(0) Z^i = 0$  in  $S[Z]/(Z^p)$ , which implies  $G_i(0) = 0$  for all  $i$ , but the kernel of the map from  $B/\text{nil } B \rightarrow S[Z]/(Z^p)$  is clearly generated by  $U$ . Then  $U \nmid G_i(U)$  provides the contradiction to  $G \neq 0 \bmod \bar{F}$ . Thus,  $\theta$  is an isomorphism, and the other maps must also be isomorphisms. In particular,  $\text{nil } B = (\text{nil } R) B$ .

*Proof of (2).* First assume the following lemma.

**LEMMA.** *There exist  $t, u \in R[X_1, X_2]$ , such that  $\bar{t} = Z - \bar{a}Z^p$  and  $\bar{u} = Z^p$  in  $R[Z - \bar{a}Z^p, Z^p]$ , and  $F(t, u) = 0$ .*

Next define  $\psi: B[Y_1] \rightarrow R[X_1, X_2]$  by  $T \rightarrow t$ ,  $U \rightarrow u$ ,  $Y_1 \rightarrow \text{lift of } y_1 = x_2 - (x_1 + ax_2)^p$ . We argue that  $\psi$  and  $\phi$  are isomorphisms. Again, the argument mimics the argument in 2.1. Reduce to the Noetherian case. Observe that  $\phi\psi$  and  $\phi\psi$  are onto endomorphisms because  $R[X_1, X_2] = \psi\phi(R[X_1, X_2]) + NR[X_1, X_2]$  with  $N$  nilpotent implies  $R[X_1, X_2] = \psi\phi(R[X_1, X_2])$ . Since  $\text{nil } B = (\text{nil } R) B$ , a similar argument implies  $\phi\psi B[Y_1] = B[Y_1]$ . Since both rings are Noetherian,  $\phi\psi$  and  $\psi\phi$  are indeed isomorphisms whence  $\psi$  and  $\phi$  are.

*Proof of Lemma.* Let  $e = F(t, u) \in N_0^s$ . Let  $F_T = \partial F / \partial T$  and  $F_U = \partial F / \partial U + 1$ . Then  $\bar{F}_T = p(T + \bar{a}U)^{p-1}$ ,  $\bar{F}_U = p\bar{a}(T + \bar{a}U)^{p-1}$ . Let  $t' = t + ae$ ,  $u' = u + \beta e$ . Then  $F(t', u') = F(t, u) + F_T ae + F_U \beta e - \beta e + e^2(\dots)$ .

*Step 1.* Let  $\alpha = 0, \beta = 1$ . Then  $F(t', u') = F_U \beta e + e^2(\dots)$ . But  $F_U$  has all coefficients in  $J = (b, c, d) \subset R$  so  $F_U(T, U) \in JR[X_0, X_1]$ .

*Step 2.* Let  $t'' = t' + \alpha' e, u'' = u' + \beta' e$ , where  $\beta' = F_U(t, u)$  and  $-\alpha'$  lifts  $\bar{a} \bar{F}_U(t, u)$ , which  $\varepsilon \bar{a} \bar{J}R[x_1, x_2] \subseteq \bar{J}[x_1, x_2]$ . Now,  $F(t'', u'') = F(t', u') + F_T(t', u') \alpha' e + F_U(t', u') \beta' e - \beta' e + (\beta e + e^2(\dots))^2 = [F_T(t', u') \alpha' + \beta' F_U(t', u')] e + e^2(\dots)$ . But, by construction, the expression in the brackets goes to 0 in  $\bar{R}[x_1, x_2]$ . Thus,  $F(t'', u'') \in eN_0 \subseteq N_0^{s+1}$ , and the claim follows since  $N_0^k = 0$ .



It is clear that (1) and (2) give the result as  $\bar{R}[Z - aZ^p, Z^p]$  is not isomorphic to  $\bar{R}[X]$ . While  $R$  does not necessarily have missing elements as  $\bar{R}$  does, the polynomial  $F(T, U)$  reflects the defect in  $\bar{R}$ .

#### 4. APPLICATION TO ALGERAS IN ONE VARIABLE

Cornelius Greither generalized Swan's result that a reduced ring  $R$  is steadfast iff  $R$  is  $p$ -seminormal for all  $p$  to a result about invertible algebras in one variable. Specifically, all invertible  $R$ -algebras in one variable are symmetric algebras iff  $R$  is  $p$ -seminormal for all  $p$ . (It was taken for granted that  $R$  is reduced in this statement.) Definitions follow.

**DEFINITION 4.1.** An  $R$ -algebra is projective if for any surjective  $R$ -algebra morphism  $b: B \rightarrow C$ , and for any  $R$ -algebra morphism  $a: A \rightarrow C$  there is an  $R$ -algebra morphism  $c: A \rightarrow B$  with  $bc = a$ . (Projective algebras are precisely the retracts of polynomial algebras.)

**Definition 4.2.** An  $R$ -algebra is invertible if there is some  $R$ -algebra  $B$  such that  $A \otimes_R B$  is  $R$ -algebra-isomorphic to some polynomial algebra over  $R$ .

**DEFINITION 4.3.** For any  $R$ -algebra  $A$ , we say that  $A$  is an  $R$ -algebra in one variable iff the following condition is met: There is a finite set  $\{X_1, \dots, X_n\} \subseteq A$  such that

- (i)  $A = R[X_1, \dots, X_n]$ ;
- (ii) for  $1 \leq i < j \leq n$  there is a polynomial relation  $F(X_i, X_j) = 0$ , where  $F \in R[X, Y]$  and the ideal generated by the coefficients of  $F$  has zero annihilator in  $R$ .

The following theorems extend Greither's results.

**THEOREM 4.1.**  $R$  has the property that all invertible  $R$ -algebra in one variable are symmetric iff  $R$  reduced has this property iff  $R$  reduced is  $p$ -seminormal for all  $p$ . ( $R_{\text{reduced}} = R/N$ , where  $N = \text{ideal of nilpotents}$ .)

*Proof.* The second equivalence has already been observed. First, suppose  $R_{\text{red}}$  has the property that all invertible  $R_{\text{red}}$  algebras are symmetric, whence  $R_{\text{red}}$  is  $p$ -seminormal for all  $p$ . Let  $A$  be an invertible  $R$ -algebra in one variable. Greither's paper contains the ingredients of the proof that  $A$  is symmetric. An outline follows. Reduce to the local case as in the proof of Theorem 2.3 and Theorem 4.6 of [4]. The idea is to construct  $R_0$  by adjoining to the image of  $Z$  in  $R$  a finite number of coefficients, so that if  $A = R[a_1 \cdots a_m]$  and  $A \otimes B \cong R[T_1, \dots, T_n]$ , then  $A_0 = R_0[a_1, \dots, a_m]$  is an

invertible  $R_0$  algebra in one variable, and  $A \cong A_0 \otimes_{R_0} R$ . One must also take the  $p$ -seminormalization of  $R_{0\text{red}}$  which can only require a finite number of steps, and lift. Localization of a  $p$ -seminormal ring is  $p$ -seminormal [10]. Thus  $A_0, R_0, R_{0\text{red}}$  are as  $A, R, R_{\text{red}}$ . If  $A_0$  is a symmetric algebra over  $R_0$ , then  $A \cong A_0 \otimes_{R_0} R \cong S(M) \otimes_{R_0} R \cong S(M \otimes_{R_0} R)$ .

From here the comments in Remark 4.7 of [4] apply, as they only use the assumption that  $R_{\text{red}}$  is  $p$ -seminormal and not that  $R$  itself is  $p$ -seminormal. To wit,  $A$  has an augmentation  $e$  since  $A \otimes_R B \cong R[T_1, \dots, T_n]$ . If  $I = \ker e$ , one shows  $I$  is  $A$ -free using  $N^n$  Pic  $R$  has no torsion when  $R_{\text{red}}$  is  $p$ -seminormal for all  $p$  [10].  $I$   $A$ -free implies  $A \cong R[T_1]$ , as in the proof of Theorem 2.3 [4].

What is new is that if  $R_{\text{red}}$  does not have the property in question, then  $R$  does not. Suppose there is an  $R_{\text{red}}$  algebra in one variable which is invertible but not projective, then  $R_{\text{red}} = \bar{R}$  is not  $p$ -seminormal for some  $p$ , so  $\bar{B} = \bar{R}[Z - \bar{a}Z^p, Z^p]$  can be constructed and then lifted to  $B = R[t, u]$ , so  $B[Y_1] = R[X_1, X_2]$ .  $B$  is clearly invertible.  $t$  is integral over  $R[u]$ , so  $B$  is an  $R$ -algebra in one variable. If  $B$  were a symmetric  $R$ -algebra,  $S(M)$ , then  $\bar{B} \cong S(M)_{\text{red}} \cong R/N \otimes S(M) \cong S(R/N \otimes_R M)$  would be symmetric.  $\bar{B} \cong \bar{R}[Z - \bar{a}Z^p, Z^p]$ . Let  $I = (Z - \bar{a}Z^p, Z^p)$ .  $I$  is clearly an augmentation ideal (i.e., a kernel of a homomorphism from  $\bar{B}$  to  $\bar{R}$ .) Thus,  $\bar{B} \cong S(M)$  implies  $\bar{B} \cong S(I/I^2)$  [12, 3.10]. Since  $I/I^2$  is generated by the image of  $Z - \bar{a}Z^p$  (check that  $Z^p \in I^2$ ),  $\bar{B}$  could be singly generated over  $\bar{R}$ , and hence isomorphic to  $\bar{R}[X_1]$  [5, 1.1]. Since this is not the case,  $B$  is not a symmetric algebra over  $R$ , and the result follows.

Greither's paper also shows that a reduced  $R$  has the property that all projective  $R$ -algebras in one variable are symmetric iff  $R$  is seminormal [4, 2.6, 3.6]. In fact, 2.6 states that if  $R_{\text{red}}$  is seminormal, then  $R$  has the property in question. We show the converse, so the following holds.

**THEOREM 4.2.**  *$R$  has the property that all projective  $R$ -algebras in one variable are symmetric iff  $R_{\text{red}}$  has this property iff  $R_{\text{red}}$  is seminormal.*

*Proof.* It only remains to show that if  $R_{\text{red}} = \bar{R}$  has a projective  $\bar{R}$ -algebra  $\bar{A}$  in one variable which is not symmetric, then  $R$  has such an algebra.  $\bar{R}$  cannot be seminormal, so there exist  $\bar{b}, \bar{c}$  so  $\bar{b}^3 = \bar{c}^2$ , but no  $\bar{a} \in \bar{R}$  so  $\bar{a}^2 = \bar{b}$  and  $\bar{a}^3 = \bar{c}$ . Let  $S = \bar{R}[\bar{a}] (\cong R[X]/(X^2 - \bar{b}, X^3 - \bar{c}))$  [10, 4.3].

From Greither's paper there exists  $\bar{A}$ , a projective  $\bar{R}$ -algebra which is not a symmetric algebra over  $\bar{R}$ . Further,  $\bar{A}$  can be chosen so that  $\bar{A}$  is a retract of  $\bar{R}[X_1, X_2]$ .  $\bar{A}$  will be described below for the interested reader. Let  $\bar{\phi}$  be the retract map, so that  $\bar{A} = \bar{\phi}(\bar{R}[X_1, X_2])$ . By Theorem 2.2,  $\bar{\phi}$  can be lifted to  $R[X_1, X_2]$ . Let  $\phi$  be the lift, and let  $A = \phi(R[X_1, X_2])$ . Clearly,  $A$  is a projective  $R$ -algebra.  $A$  is not a symmetric algebra over  $R$ , since  $A \otimes_R \bar{R} = \bar{A}$  is not a symmetric algebra over  $\bar{R}$ .

The argument that  $A$  is an  $R$ -algebra in one variable is essentially Greither's. If the coefficients of elements in  $\text{Ker } \phi$  do not generate all of  $R$ , then  $\text{Ker } \phi \subseteq M[X_1, X_2]$ . In this case,  $A \otimes_R R/M$  would be isomorphic to a polynomial ring in two variables, which is impossible for the particular example. Thus, the ideal of coefficients is all of  $R$ . Only a finite number of coefficients are required to generate  $R$ , so only a finite number of polynomials,  $h_1, \dots, h_n$  are required. Then  $h^* = h_1 + X_1^N h_2 + \dots + X_1^{(n-1)N} h_n$  has coefficients which generate the unit ideal for  $N$  large enough to prevent "overlaps." Since  $A = R[\phi(X_1), \phi(X_2)]$ , and  $\phi(X_1), \phi(X_2)$  satisfy  $h^* = 0$ , the result follows.

**THE EXAMPLE  $\bar{A}$ .** Let  $\bar{A} = \bar{R}[Z - \bar{a}Z^2, \bar{u}] \subseteq S[Z]$ . Recall  $S = R[\bar{a}]$ .  $\bar{u}$  is chosen in  $\bar{a}^4 S[Z]$  so that  $Z = \psi(Z - \bar{a}Z^2) + \bar{u}$ . If  $\bar{i} = Z - \bar{a}Z^2$ , then  $S[Z] = S[\bar{i}, \bar{u}]$ . Since  $S[[\bar{i}]] = S[[Z]]$ , it is easy to find  $\bar{u}$ . It can be shown that  $\bar{a}^i Z^j \in \bar{A}$  for all  $i \geq 2, j \geq 1$ . In fact, if  $R$  has characteristic 2,  $\bar{A}$  is our old friend  $\bar{B} = \bar{R}[Z - \bar{a}Z^2, Z^2]$ . If  $\bar{u} = \bar{a}^2 f$ , define  $p: \bar{R}[X_1, X_2] \rightarrow \bar{A}$ , by sending  $X_1 \rightarrow \bar{i}$  and  $X_2 \rightarrow f$ . Define  $j: S[Z] \rightarrow \bar{R}[X_1, X_2]$  by sending  $Z$  to  $\psi(X_1) + \bar{a}^2 X_2$ , and then restrict  $j$  to  $\bar{A}$ . Greither's paper shows  $pj$  is the identity on  $\bar{A}$ . The paper also shows that if  $\bar{A}$  were a symmetric algebra over  $\bar{R}$ , then  $\bar{A}$  would be isomorphic to  $\bar{R}[X]$  which is not the case.

We also obtain the following generalization of 4.3 of [4].

**THEOREM 4.3.** *If  $R$  is a ring such that  $R_{\text{red}}$  is  $p$ -seminormal for all  $p$  but not seminormal, then there exists a projective algebra in one variable which is not invertible.*

*Proof.* By 4.2,  $R$  has a projective algebra  $A$  in one variable which is not symmetric. If it were invertible, then 4.1 would be contradicted.

We summarize the results.

$R$  is a commutative ring with 1.

$A$  is an algebra in 1 variable over  $R$ .

(1)  $R$  is reduced.

(2)  $R$  is  $p$ -seminormal for all  $p$ .

(3)  $R$  is seminormal.

(4)  $A$  is symmetric.

(5)  $A$  is projective.

(6)  $A$  is invertible.

(6)  $\Rightarrow$  (5); (4) & (5)  $\Rightarrow$  (6); (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1);

$R_{\text{red}}$  is (2)  $\Leftrightarrow$  [(4)  $\Leftrightarrow$  (6)];

$R_{\text{red}}$  is (3)  $\Leftrightarrow$  [(4)  $\Leftrightarrow$  (5)], so [(4)  $\Leftrightarrow$  (5)  $\Leftrightarrow$  (6)],

$R_{\text{red}}$  is (2) not (3)  $\Rightarrow \exists A$  (5) not (6).

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## REFERENCES

1. S. ABHYANKAR, P. EAKIN, AND W. HEINZER, On the uniqueness of the coefficient ring in polynomial ring, *J. Algebra* **23** (1972), 310–342.
2. P. EAKIN AND W. HEINZER, “A Cancellation Problem for Rings,” *Lecture Notes in Mathematics* 311, pp. 61–77, Springer-Verlag, Heidelberg/Berlin/New York, 1972.
3. P. EAKIN AND K. KUBOTA, A note on the uniqueness of rings of coefficients in polynomial rings, *Proc. Amer. Math. Soc.* **32** (1972), 333–341.
4. CORNELIUS GREITHER, Seminormality, projective algebras, and invertible algebras, *J. Algebra* **70** (1981), 316–338.
5. E. HAMANN, On the  $R$ -invariance of  $R[X]$ , *J. Algebra* **35** (1975), 1–16.
6. M. HOCHSTER, Nonuniqueness of coefficient rings in a polynomial ring, *Proc. Amer. Math. Soc.* **34** (1972), 81–82.
7. MIYANISHI, Simple birational extensions of a polynomial ring  $k[x, y]$ , *Osaka J. Math* **15** (1978), 663–677.
8. M. NAGATA, “Local Rings,” *Tracts in Pure and Applied Mathematics* No. 13, Interscience, New York, 1962.
9. J. SILVESTER, “Introduction to Algebraic  $K$  – Theory,” Chapman and Hall, London and New York,
10. R. SWAN, On seminormality, *J. Algebra* **67** (1980), 210–229.
11. C. TRAVERSO, Seminormality and Picard groups, *Pisa Scuola Norm. Sup. Ann. Ser. 3*, **24** (1970), 585–595.
12. D. WRIGHT, “Algebras Which Resemble Symmetric Algebras,” Thesis, Columbia University, 1975.